IIT JAM Physics

Vectors (5 Lectures)

- Vector algebra.
- Scalar and vector products of two, three and four vectors.
- Polar and Axial vectors
- Derivatives of a vector with respect to a parameter.
- Gradient, divergence and curl of vectors fields.

Vector Algebra

- Addition & Multiplication of Vector
- Dot & Cross product of two vectors
- Dot & Cross product of three vectors
- Dot & Cross product of Four vectors
- Different Conditions in vectors

- **Vector and Scalar quantities**

If you walk 4 miles due north and then 3 miles due east (Fig. 1.1), you will have gone a total of 7 miles, but you're *not* 7 miles from where you set out—you're only 5. We need an arithmetic to describe quantities like this, which evidently do not add in the ordinary way. The reason they don't, of course, is that **displacements** (straight line segments going from one point to another) have *direction* as well as *magnitude* (length), and it is essential to take both into account when you combine them. Such objects are called **vectors**: velocity, acceleration, force and momentum are other examples. By contrast, quantities that have magnitude but no direction are called **scalars**: examples include mass, charge, density,

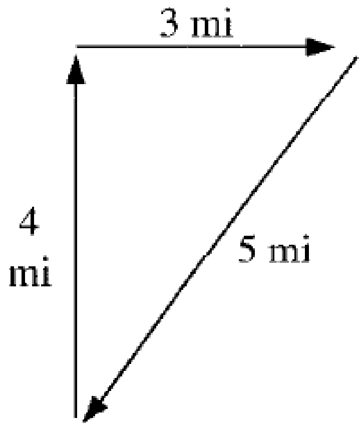


Figure 1.1

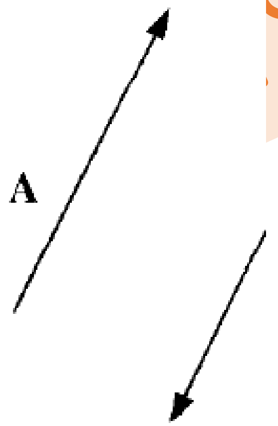


Figure 1.2

Minus **A** (−**A**) is a vector with the same magnitude as **A** but of opposite direction

Note that vectors have magnitude and direction but *not location*

• **Polar & Axial Vectors**

Polar vectors:

The vectors associated with a linear directional effect are called polar vectors.

The examples of polar vectors are force, acceleration, linear velocity, linear momentum

Axial vectors

The vectors associated with rotation about an axis are called axial vectors.

The examples of axial vectors are: torque, angular velocity, angular momentum etc.

Types of Vector

- **Zero Vector**: ($\vec{0}$) A vector whose initial and terminal points coincides
- **Unit Vector ($\hat{a}$) :** A vector whose magnitude is unity
- **Co-initial Vectors:** Two or more vector have same initial points
- **Collinear Vectors**: Two or more vectors are said to be collinear if they are parallel to the same line irrespective of magnitude and direction.
- **Equal Vector:** They have same magnitude and direction

- **Vector Operations:** We define four vector operations: addition and three kinds of multiplication.

(i) Addition of two vectors. Place the tail of **B** at the head of **A**; the sum, **A** + **B**, is the vector from the tail of **A** to the head of **B** (Fig. 1.3). (This rule generalizes the obvious procedure for combining two displacements.) Addition is *commutative:*

- **Commutative**

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$

- **Associative**

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$

To subtract a vector (Fig. 1.4), add its opposite:

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})$$

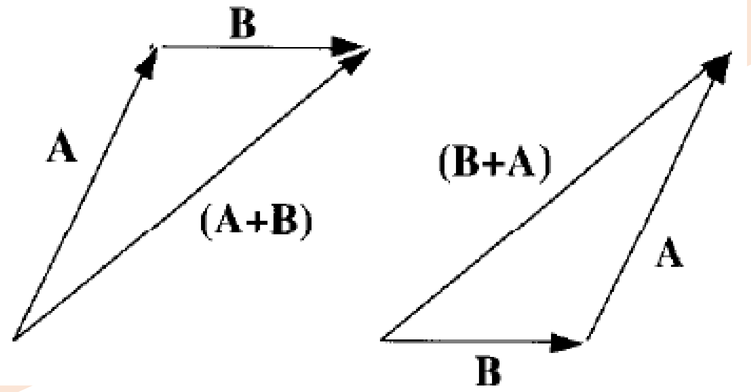


Figure 1.3

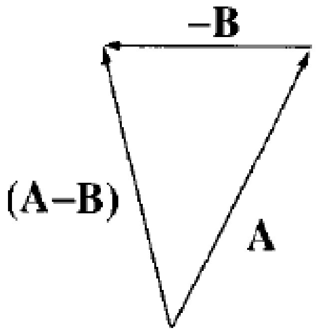


Figure 1.4

(ii) Multiplication by a scalar. Multiplication of a vector by a positive scalar a multiplies the *magnitude* but leaves the direction unchanged (Fig. 1.5). (If a is negative, the direction is reversed.) Scalar multiplication is *distributive:*

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}.$$

(iii) Dot product of two vectors. The dot product of two vectors is defined by

$$\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta, \qquad (1.1)$$

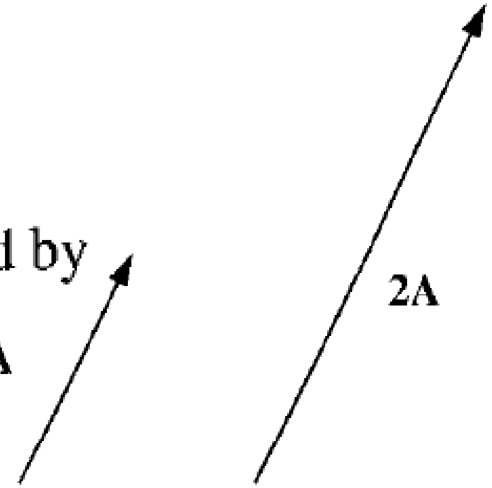


Figure 1.5

where θ is the angle they form when placed tail-to-tail (Fig. 1.6). Note that $\mathbf{A} \cdot \mathbf{B}$ is itself a *scalar* (hence the alternative name **scalar product**). The dot product is *commutative*,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

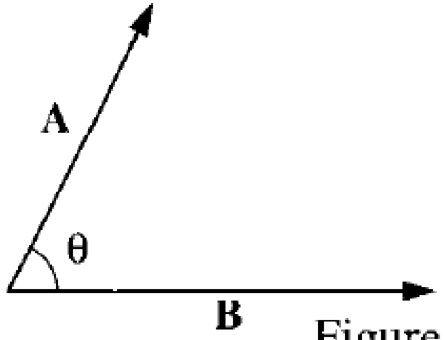


Figure 1.6

- **Distributive Law**

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \qquad (1.2)$$

Geometrically, $\mathbf{A} \cdot \mathbf{B}$ is the product of A times the projection of $\mathbf{B}$ along $\mathbf{A}$ (or the product of B times the projection of $\mathbf{A}$ along $\mathbf{B}$). If the two vectors are parallel, then $\mathbf{A} \cdot \mathbf{B} = AB$. In particular, for any vector $\mathbf{A}$,

$$\mathbf{A} \cdot \mathbf{A} = A^2. \qquad (1.3)$$

If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A} \cdot \mathbf{B} = 0$

- Ex Let $\mathbf{C} = \mathbf{A} \quad \mathbf{B}$ (Fig. 1.7), and calculate the dot product of $\mathbf{C}$ with itself.
- Soln $\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B}.$

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$ This is the **law of cosines**

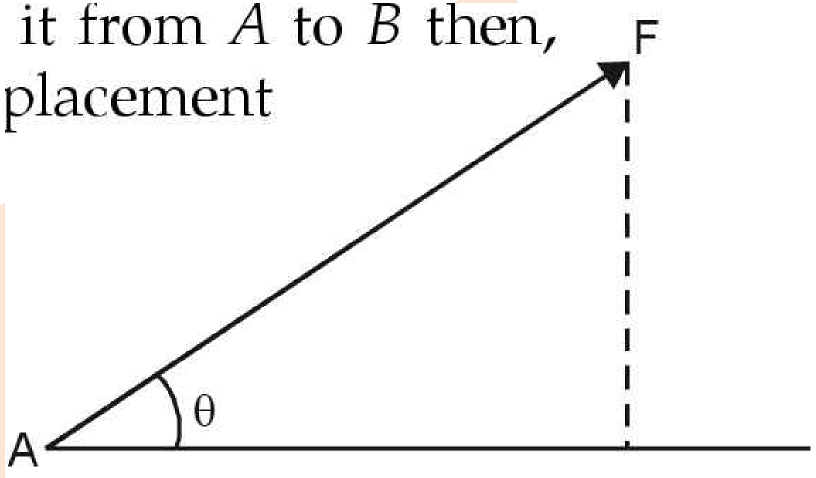
• Work done as scalar Product

If a constant force F acting on a particle displaces it from A to B then,

Work done = (component of F along AB). Displacement

$$= \text{F} \cos \theta . \; AB$$

$$= \vec{F} \cdot \vec{AB}$$



Work done = Force . Displacement

USEFUL RESULTS

$\hat{i} \cdot \hat{i} = (1)\,(1) \cos 0° = 1$ Similarly, $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = (1)\,(1) \cos 90° = 0$ Similarly, $\hat{j} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{i} = 0$

- . **(iv) Cross product of two vectors.** The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta \, \hat{\mathbf{n}}, \qquad (1.4)$$

where $\hat{\mathbf{n}}$ is a **unit vector** (vector of length 1) pointing perpendicular to the plane of **A** and **B**. (I shall use a hat (ˆ) to designate unit vectors.) Of course, there are *two* directions perpendicular to any plane: "in" and "out." The ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{\mathbf{n}}$. (In Fig. 1.8 $\mathbf{A} \times \mathbf{B}$ points *into* the page; $\mathbf{B} \times \mathbf{A}$ points *out* of the page.) Note that $\mathbf{A} \times \mathbf{B}$ is itself a *vector* (hence the alternative name **vector product**). The cross product is *distributive*,

- Distributive Law

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}) \qquad (1.5)$$

but *not commutative*. In fact,

$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B}) \qquad (1.6)$$

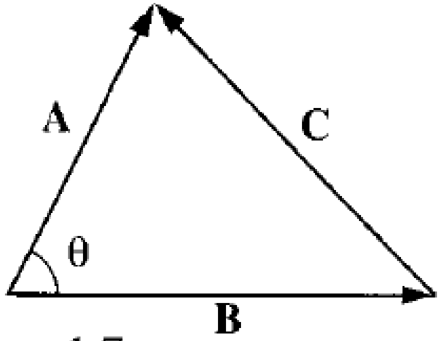


Figure 1.7

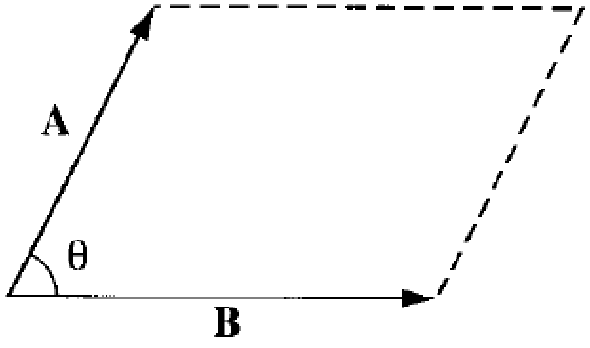


Figure 1.8

• **Note** Geometrically, $|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram generated by **A** and **B** (Fig. 1.8). If two vectors are parallel, their cross product is zero. In particular,

$$\mathbf{A} \times \mathbf{A} = 0 \qquad \text{for any vector } \mathbf{A}.$$

Useful results

Since $\hat{i}, \hat{j}, \hat{k}$ are three mutually perpendicular unit vectors, then

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}$$

$$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$$

$$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

and

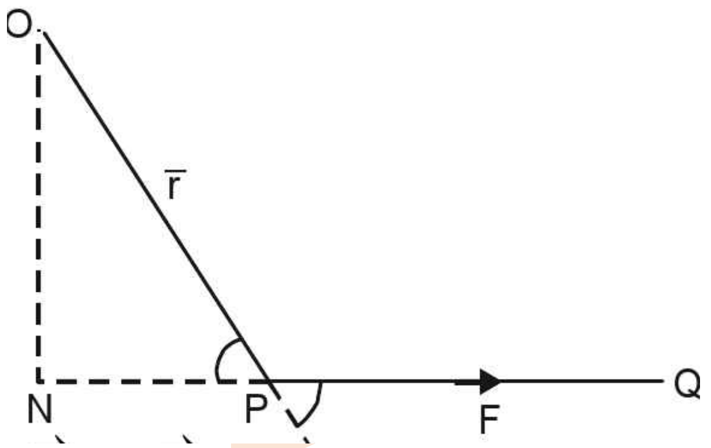
$$\hat{j} \times \hat{i} = -\hat{i} \times \hat{j}$$

$$\hat{k} \times \hat{j} = -\hat{j} \times \hat{k}$$

$$\hat{i} \times \hat{k} = -\hat{k} \times \hat{i}$$

• Example of cross product: Torque & Angular Velocity

MOMENT OF A FORCE



Let a force F $(\overrightarrow{PQ})$ act at a point P.

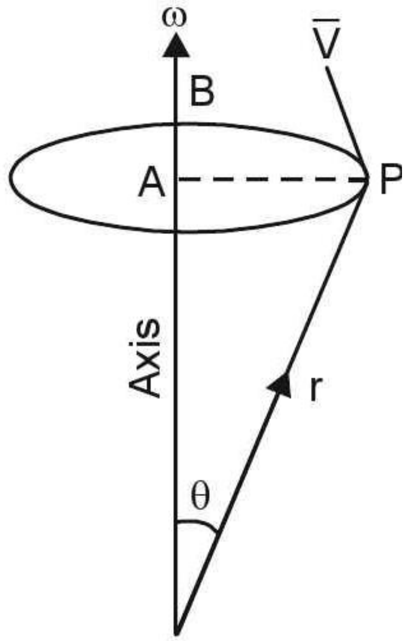
Moment of $\vec{F}$ about O

= Product of force F and perpendicular distance (ON. $\hat{\eta}$)

$= (PQ)\,(ON)(\hat{\eta}) = (PQ)\,(OP)\sin\theta\,(\hat{\eta}) = OP \times PQ$

$\Rightarrow \quad \vec{M} = \vec{r} \times \vec{F}$

Let a rigid body be rotating about the axis OA with the angular velocity ω which is a vector and its magnitude is ω radians per second and its direction is parallel to the axis of rotation OA.



Let P be any point on the body such that $\overline{OP} = \vec{r}$ and $\angle AOP = \theta$ and $AP \perp OA$. Let the velocity of P be V.

Let $\hat{\eta}$ be a unit vector perpendicular to $\vec{\omega}$ and $\vec{r}$.

$\vec{\omega} \times \vec{r} = (\omega\, r \sin\theta)\,\hat{\eta} = (\omega\, AP)\,\hat{\eta} = (\text{Speed of } P)\,\hat{\eta}$

= Velocity of $P \perp$ to $\vec{\omega}$ and r

Hence $\boxed{\vec{V} = \vec{\omega} \times \vec{r}}$

• Vector Algebra: Component Form

In the previous section I defined the four vector operations (addition, scalar multiplication, dot product, and cross product) in "abstract" form—that is, without reference to any particular coordinate system. In practice, it is often easier to set up Cartesian coordinates x, y, z and work with vector "components." Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ be unit vectors parallel to the x, y, and z axes, respectively (Fig. 1.9(a)). An arbitrary vector $\mathbf{A}$ can be expanded in terms of these **basis vectors** (Fig. 1.9(b)):

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}.$$

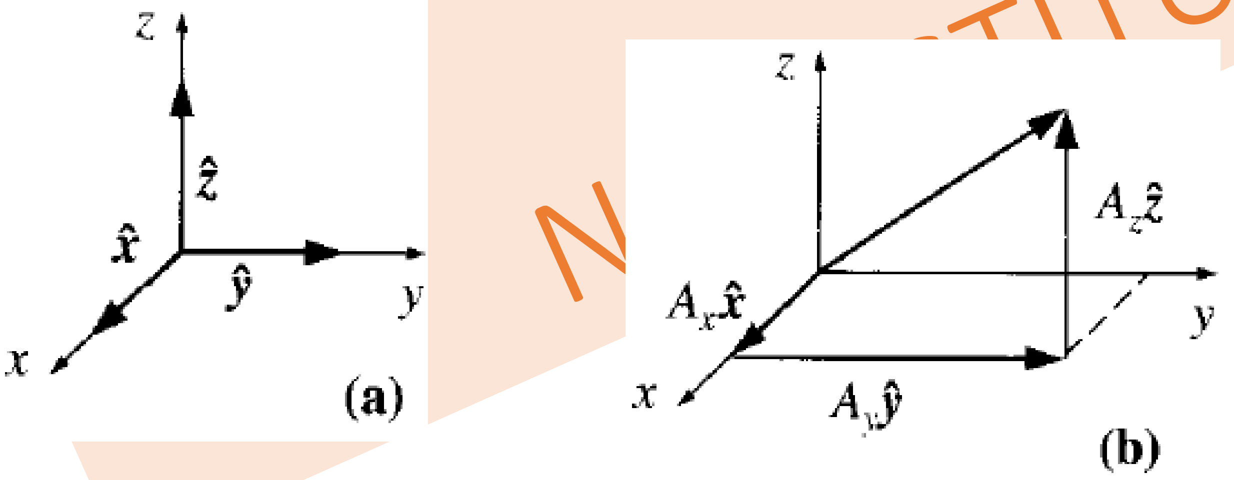


Figure 1.9

- > The numbers A_x, A_y, and A_z, are called **components** of **A**; geometrically, they are the projections of **A** along the three coordinate axes. We can now reformulate each of the four vector operations as a rule for manipulating components:

$$\mathbf{A}+\mathbf{B}=(A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}})+(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}})=(A_x+B_x)\hat{\mathbf{x}}+(A_y+B_y)\hat{\mathbf{y}}+(A_z+B_z)\hat{\mathbf{z}}. \quad (1.7)$$

(i) Rule: *To add vectors, add like components*

$$a\mathbf{A}=(aA_x)\hat{\mathbf{x}}+(aA_y)\hat{\mathbf{y}}+(aA_z)\hat{\mathbf{z}}. \quad (1.8)$$

(ii) Rule: *To multiply by a scalar, multiply each component.*

Because $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular unit vectors,

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1; \quad \hat{\mathbf{x}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{z}}=0. \quad (1.9)$$

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}})\cdot(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}})\\ &= A_xB_x+A_yB_y+A_zB_z \quad (1.10)\end{aligned}$$

- **(iii) Rule:** *To calculate the dot product, multiply like components, and add.*

In particular,

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2, \qquad A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.11}$$

(This is, if you like, the three-dimensional generalization of the Pythagorean theorem.) Note that the dot product of $\mathbf{A}$ with any *unit* vector is the component of $\mathbf{A}$ along that direction (thus $\mathbf{A} \cdot \hat{\mathbf{x}} = A_x$, $\mathbf{A} \cdot \hat{\mathbf{y}} = A_y$, and $\mathbf{A} \cdot \hat{\mathbf{z}} = A_z$).

$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0, & \hat{\mathbf{y}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} &= \hat{\mathbf{x}}, \\ \hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} &= \hat{\mathbf{z}}, & \hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} &= \hat{\mathbf{y}}. \end{aligned} \tag{1.12}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}. \end{aligned} \tag{1.13}$$

- **Determinant Form**

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

(iv) Rule: *To calculate the cross product, form the determinant whose first row is* $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, *whose second row is* $\mathbf{A}$ *(in component form), and whose third row is* $\mathbf{B}$.

• Ex Find the angle between the face diagonals of a cube.

Solution: We might as well use a cube of side 1, and place it as shown in Fig. 1.10, with one corner at the origin. The face diagonals **A** and **B** are

$$\mathbf{A} = 1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 0\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}.$$

So, in component form,

$$\mathbf{A} \cdot \mathbf{B} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1.$$

On the other hand, in "abstract" form,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta = \sqrt{2}\sqrt{2}\cos\theta = 2\cos\theta.$$

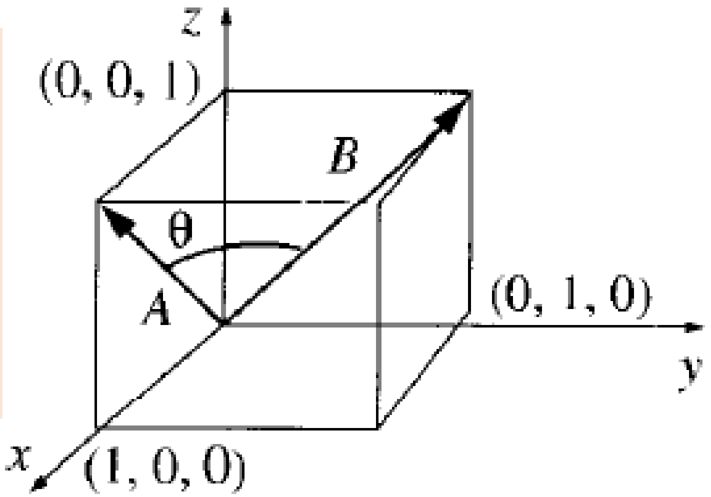


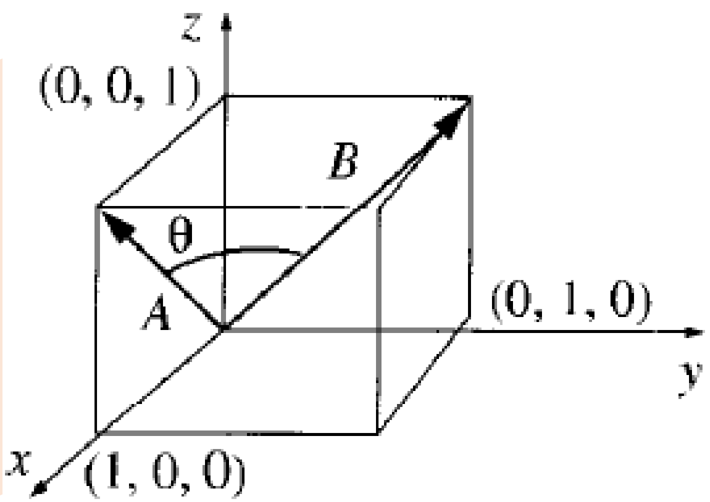
Figure 1.10

Therefore,

$$\cos\theta = 1/2, \quad \text{or} \quad \theta = 60°.$$

Of course, you can get the answer more easily by drawing in a diagonal across the top of the cube, completing the equilateral triangle. But in cases where the geometry is not so simple, this device of comparing the abstract and component forms of the dot product can be a very efficient means of finding angles.

- **EX** Find the angle between the body diagonals of a cube.
- **Ex** Write down the coordinate body diagonals, face diagonals and corners of cube assuming one corner is origin.



• Triple products: Dot & Cross Product of three vectors

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a *triple* product.

(i) Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Geometrically, $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ is the volume of the parallelepiped generated by **A**, **B**, and **C**, since $|\mathbf{B} \times \mathbf{C}|$ is the area of the base, and $|A \cos\theta|$ is the altitude (Fig. 1.12). Evidently,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \tag{1.15}$$

for they all correspond to the same figure. Note that "alphabetical" order is preserved—in view of Eq. 1.6, the "nonalphabetical" triple products, have the opposite sign.

$$\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}),$$

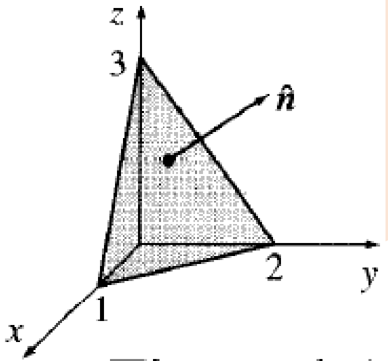


Figure 1.11

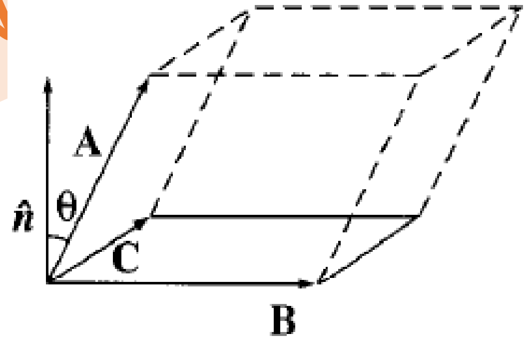


Figure 1.12

• **In determinant form**

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \qquad (1.16)$$

Note that the dot and cross can be interchanged: $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ (this follows immediately from Eq. 1.15); however, the placement of the parentheses is critical: $(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$ is a meaningless expression—you can't make a cross product from a *scalar* and a vector.

(ii) Vector triple product: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The vector triple product can be simplified by the so-called **BAC-CAB** rule:

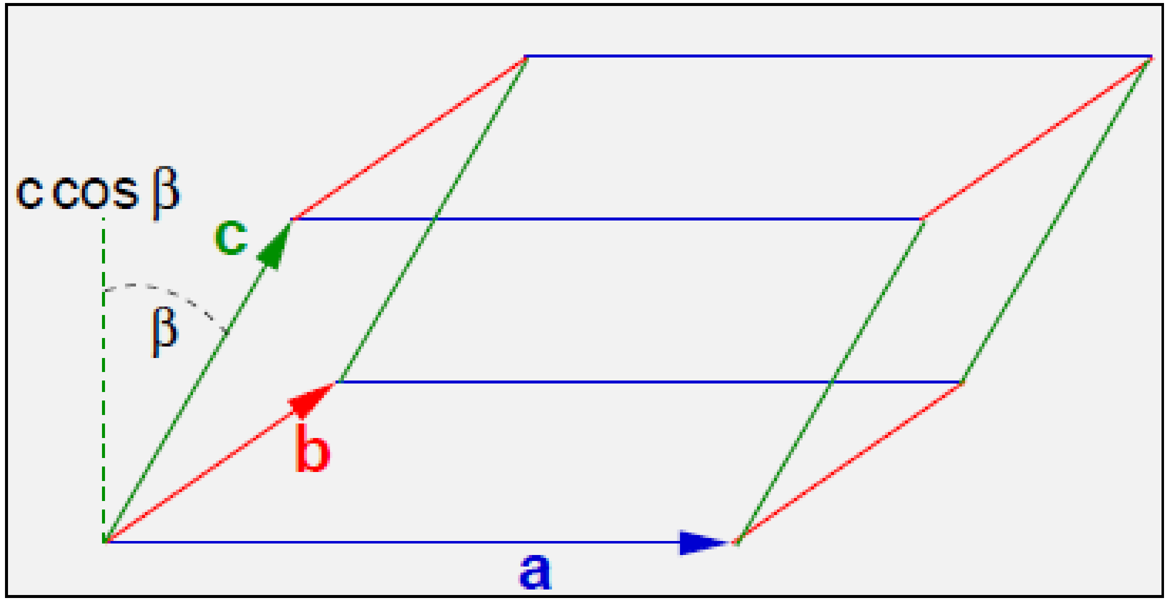
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \qquad (1.17)$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = -\mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{B}(\mathbf{A}\cdot\mathbf{C})$$

is an entirely different vector. Incidentally, all *higher* vector products can be similarly reduced, often by repeated application of Eq. 1.17, so it is never necessary for an expression to contain more than one cross product in any term. For instance,

• Geometrical interpretation of scalar triple product

- The scalar triple product gives the volume of the parallelopiped whose sides are represented by the vectors **a**, **b**, and **c**.



- Vector product $(\mathbf{a} \times \mathbf{b})$ has magnitude equal to the area of the base direction perpendicular to the base.

- The *component* of **c** in this direction is equal to the height of the parallelopiped

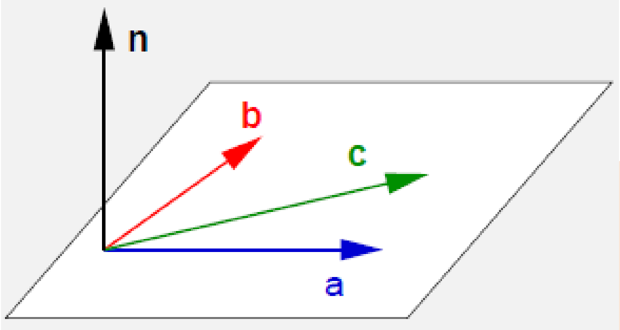
Hence

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \text{volume of parallelopied}$$

- **Linearly dependent vectors**

If the scalar triple product of three vectors

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$



- **Theorem:** *If* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},\ \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},\ \textit{and}\ \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k},$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- **Dot & Cross properties**

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \textit{Since dot and cross can be interchanged.} \\
&= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}, \textit{Since dot product is commutative.} \\
&= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}), \textit{Since dot and cross can be interchanged.} \\
&= (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}, \textit{Since dot product is commutative.} \\
&= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \textit{Since dot and cross can be interchanged.}
\end{aligned}$$

- **Example to calculate the Vol of Tetrahedron and Coplanar condition**

Example 7. *Find the volume of tetrahedron having vertices*

$$(-\hat{j}-\hat{k}),\quad (4\hat{i}+5\hat{j}+q\hat{k}),\quad (3\hat{i}+9\hat{j}+4\hat{k}) \text{ and } 4(-\hat{i}+\hat{j}+\hat{k}).$$

Also find the value of q for which these four points are coplanar.

(*Nagpur University, Summer 2004, 2003, 2002*)

Solution. Let $\vec{A} = -\hat{j}-\hat{k},\ \bar{B} = 4\hat{i}+5\hat{j}+q\hat{k},\ \bar{C} = 3\hat{i}+9\hat{j}+4\hat{k},\ \bar{D} = 4(-\hat{i}+\hat{j}+\hat{k})$

$$\overline{AB} = \vec{B}-\vec{A} = 4\hat{i}+5\hat{j}+q\hat{k}-(-\hat{j}-\hat{k}) = 4\hat{i}+6\hat{j}+(q+1)\hat{k}$$

$$\overline{AC} = \vec{C}-\vec{A} = (3\hat{i}+9\hat{j}+4\hat{k})-(-\hat{j}-\hat{k}) = 3\hat{i}+10\hat{j}+5\hat{k}$$

$$\overline{AD} = \vec{D}-\vec{A} = 4(-\hat{i}+\hat{j}+\hat{k})-(-\hat{j}-\hat{k}) = -4\hat{i}+5\hat{j}+5\hat{k}$$

$$\text{Volume of the tetrahedron} = \frac{1}{6}[\overline{AB}\ \overline{AC}\ \overline{AD}]$$

$$= \frac{1}{6}\begin{vmatrix} 4 & 6 & q+1 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} = \frac{1}{6}\{4(50-25)-6(15+20)+(q+1)(15+40)\}$$

$$= \frac{1}{6}\{100-210+55(q+1)\} = \frac{1}{6}(-110+55+55q)$$

$$= \frac{1}{6}(-55+55q) = \frac{55}{6}(q-1)$$

If four points A, B, C and D are coplanar, then $(\overline{AB}\ \overline{AC}\ \overline{AD}) = 0$

i.e., Volume of the tetrahedron = 0

$$\Rightarrow \quad \frac{55}{6}(q-1) = 0 \quad \Rightarrow \quad q = 1$$

- **Problems**

1. Show that $\vec{a} \times (\vec{b} \times \vec{a}) = (\vec{a} \times \vec{b}) \times \vec{a}$
2. Write the correct answer

   (*a*) $(\vec{A} \times \vec{B}) \times \vec{C}$ lies in the plane of

   (*i*) $\vec{A}$ and $\vec{B}$ (*ii*) $\vec{B}$ and $\vec{C}$ (*iii*) $\vec{C}$ and $\vec{A}$ **Ans.** (*ii*)

   (*b*) The value of $\vec{a} \cdot (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c})$ is

   (*i*) Zero (*ii*) $[\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}]$ (*iii*) $[\vec{a}, \vec{b}, \vec{c}]$ (*iv*) None of these

   Ans. (*ii*)

- **Notes (1)** Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?
- **(2)**

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Notes on Triple Product

Notation: For any three vectors $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$, the scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is read as box a, b, c . For this reason and also because the absolute value of a scalar triple product represents the volume of a box (rectangular parallelepiped),a scalar triple product is also called a box product.

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = [\mathbf{b}, \mathbf{c}, \mathbf{a}]$$

$$[\mathbf{b}, \mathbf{c}, \mathbf{a}] = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b} = [\mathbf{c}, \mathbf{a}, \mathbf{b}].$$

In other words, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}]$; that is, if the three vectors are permuted in the same cyclic order, the value of the scalar triple product remains the same.

If any two vectors are interchanged in their position in a scalar triple product, then the value of the scalar triple product is (-1) times the original value. More explicitly,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}].$$

- **Summary**

- The scalar triple product is unchanged under a circular shift of its three operands (a, b, c):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

- Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

- Swapping any two of the three operands negates the triple product. This follows from the circular-shift property and the anticommutativity of the cross product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}).$$

- The scalar triple product can also be understood as the determinant of the 3×3 matrix that has the three vectors either as its rows or its columns

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det\left(\mathbf{a}, \mathbf{b}, \mathbf{c}\right).$$

- The scalar triple product can also be understood as the determinant of the 3×3 matrix that has the three vectors either as its rows or its columns

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

- If the scalar triple product is equal to zero, then the three vectors a, b, and c are coplanar, since the parallelepiped defined by them would be flat and have no volume.
- If any two vectors in the scalar triple product are equal, then its value is zero:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{a} = 0.$$

- **Ex** *Let* $\alpha = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})$. *Show that*

$$\alpha = (\mathbf{c} \cdot \mathbf{a})|\mathbf{a}|^2 - ((\mathbf{c} \cdot \mathbf{a}))(\mathbf{a} \cdot \mathbf{b}).$$

Evaluate α *when* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are unit vectors with* $\mathbf{b}$ *and* $\mathbf{c}$ *perpendicular, and the angles between* $\mathbf{a}$ *and* $\mathbf{b}$, *and between* $\mathbf{a}$ *and* $\mathbf{c}$, *are both* $\pi/3$.

Examples

Example 2.6. *If* $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{i}+m\mathbf{j}+4\mathbf{k}$ *are coplanar, find the value of* m .

Proof. Since the given three vectors are coplanar, we have

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0 \implies m = -3$$

Example 2.7. *Show that the four points (6, 7, 0), (16, 19, 4), (0, 3, 6), (2, 5,10) lie on a same plane.*

Proof. Let A = (6, 7, 0), B = (16, 19, 4), C = (0, 3, 6), D = (2, 5,10) . To show that the four points A, B, C, D lie on a plane, we have to prove that the three vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (16\mathbf{i} - 19\mathbf{j} - 4\mathbf{k}) - (6\mathbf{i} - 7\mathbf{j}) = (10\mathbf{i} - 12j - 4\mathbf{k})$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-6\mathbf{i} + 10\mathbf{j} - 6\mathbf{k}) \text{ and } \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-4\mathbf{i} + 2\mathbf{j} + 10\mathbf{k}).$$

$$[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix} = 0.$$

• EXERCISE

1. Determine λ such that

   $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} - 4\hat{k}$, and $\bar{c} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are coplanar. **Ans.** $\lambda = 5/3$

2. Show that the four points

   $-6\hat{i} + 3\hat{j} + 2\hat{k}$, $3\hat{i} - 2\hat{j} + 4\hat{k}$, $5\hat{i} + 7\hat{j} + 3\hat{k}$ and $-13\hat{i} + 17\hat{j} - \hat{k}$ are coplanar.

3. Find the constant a such that the vectors

   $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$, and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar. **Ans.** -4

4. Prove that four points

   $4\hat{i} + 5\hat{j} + \hat{k}$, $-(\hat{j} + \hat{k})$, $3\hat{i} + 9\hat{j} + 4\hat{k}$, $4(-\hat{i} + \hat{j} + \hat{k})$ are coplanar.

5. If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, show that

$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \end{vmatrix} = 0$$

Dot & Cross product of four vectors

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C});\\ \mathbf{A}\times(\mathbf{B}\times(\mathbf{C}\times\mathbf{D})) &= \mathbf{B}(\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D}))-(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\times\mathbf{D}).\end{aligned}$$

- Ex

Prove the **BAC-CAB** rule by writing out both sides in component form.

- Ex **Prove that**

$$[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]+[\mathbf{B}\times(\mathbf{C}\times\mathbf{A})]+[\mathbf{C}\times(\mathbf{A}\times\mathbf{B})]=0.$$

- **Prob**

1. If $\vec{a} = 2i + 3j - k$, $\vec{b} = -i + 2j - 4k$, $\vec{c} = i + j + k$, find $(\vec{a}\times\vec{b})\,.\,(\vec{a}\times\vec{c})$. **Ans. –74**

2. Prove that $(\vec{a}\times\vec{b})\,.\,(\vec{a}\times\vec{c}) = a^2(\vec{b}\,.\,\vec{c}) - (\vec{a}\,.\,\bar{b})(\vec{a}\,.\,\vec{c})$.

- **EXERCISE**

1. $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = \vec{c}\,(\vec{a}\ \vec{b}\ \vec{c})$ when $(\vec{a}\ \vec{b}\ \vec{c})$ stands for scalar triple product.
2. $[\vec{b} \times \vec{c},\quad \vec{c} \times \vec{a},\quad \vec{a} \times \vec{b})] = [\vec{a}\ \vec{b}\ \vec{c}]^2$
3. $\vec{d}\,[\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\}] = [(\vec{b} \cdot \vec{d})[\vec{a} \cdot (\vec{c} \times \vec{d})]$
4. $\vec{a}\,[\vec{a} \times [\vec{a} \times (\vec{a} \times \vec{b})] = a^2\,(\vec{b} \times \vec{a})$
5. $[(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})] \cdot \vec{d} = (\vec{a} \cdot \vec{d})\,[\vec{a}\ \vec{b}\ \vec{c}]$
6. $2a^2 = \left|\vec{a} \times \hat{i}\right|^2 + \left|\vec{a} \times \hat{j}\right|^2 + \left|\vec{a} \times \hat{k}\right|^2$
7. $\vec{a} \times \vec{b} = [(\hat{i} \times \vec{a}) \cdot \bar{b}]\,\hat{i} + [(\hat{j} \times \vec{a}) \cdot \vec{b}]\,\hat{j} + [(\hat{k} \times \vec{a}) \cdot \vec{b}]\,\hat{k}$
8. $\vec{p} \times [(\vec{a} \times \vec{q}) \times (\vec{b} \times \vec{r})] + \vec{q} \times [(\vec{a} \times \vec{r}) \times (\vec{b} \times \vec{p})] + \vec{r} \times [(\vec{a} \times \vec{p}) \times (\vec{b} \times \vec{q})] = 0$

LEC 5

- Position Vector & Coordinate system
- Differential Calculus :Derivatives of a vector with respect to a parameter
- Gradient
- Divergence
- Curl

• Position, Displacement and Separation Vector

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin (Fig. 1.13) is called the **position vector**:

$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}.$$

Its magnitude,

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad (1.20)$$

is the distance from the origin, and unit vector pointing radially outward

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \qquad (1.21)$$

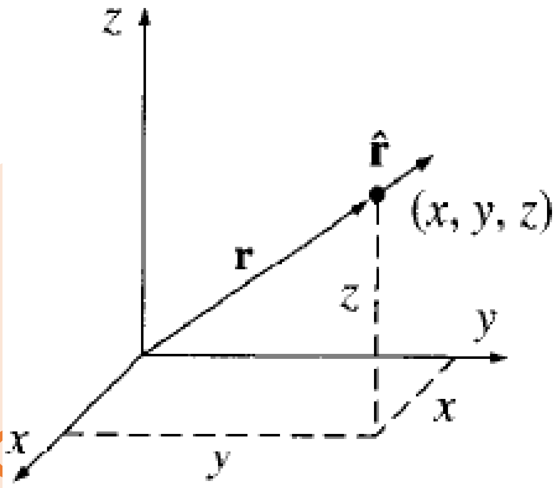


Figure 1.13

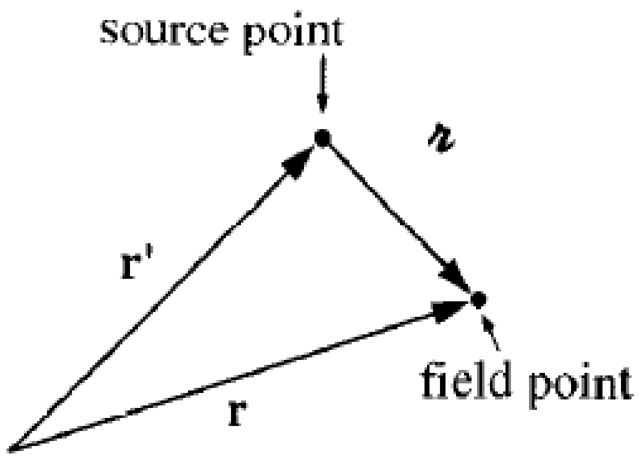


Figure 1.14

The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}.$$

(We could call this $d\mathbf{r}$, since that's what it *is*, but it is useful to reserve a special letter for infinitesimal displacements.)

- **Note**

In electrodynamics one frequently encounters problems involving *two* points—typically, a **source point**, $\mathbf{r}'$, where an electric charge is located, and a **field point**, $\mathbf{r}$, at which you are calculating the electric or magnetic field (Fig. 1.14). It pays to adopt right from the start some short-hand notation for the **separation vector** from the source point to the field point. I shall use for this purpose the script letter $\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'.$$

Its magnitude is

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$$

and a unit vector in the direction from $\mathbf{r}'$ to $\mathbf{r}$ is

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

In Cartesian coordinates,

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}},$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

(from which you can begin to appreciate the advantage of the script-$\mathscr{r}$ notation).

- **Position Vector in Different Coordinates System**
- Generally we have 3 types of Coordinates known as Curvilinear Coordinates
- Cartesian Coordinates System
- Spherical Polar Coordinates System
- Cylindrical Coordinates System

• Spherical Polar Coordinates System: Position Vector

The spherical polar coordinates (r, θ, ϕ) of a point P are defined in Fig. 1.36; r is the distance from the origin (the magnitude of the position vector), θ (the angle down from the z axis) is called the **polar angle**, and ϕ (the angle around from the x axis) is the **azimuthal angle**. Their relation to Cartesian coordinates (x, y, z) can be read from the figure:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Figure 1.36 also shows three unit vectors, $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$), and any vector $\mathbf{A}$ can be expressed in terms of them in the usual way:

$$\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}.$$

A_r, A_θ, and A_ϕ are the radial, polar, and azimuthal components of $\mathbf{A}$. In terms of the

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},\\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}},\\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}},
\end{aligned}$$

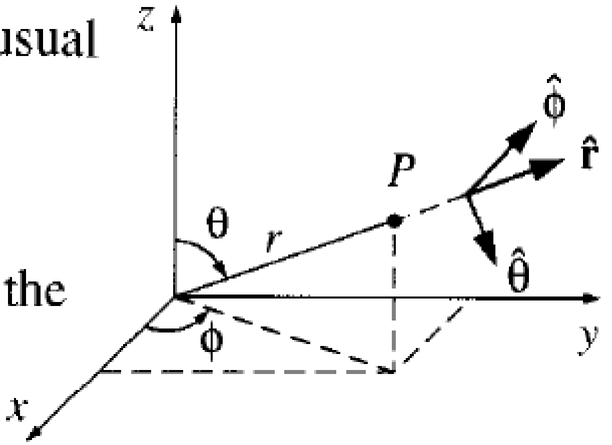


Figure 1.36

• Infinitesimal Displacement

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}.$$

- **Cylindrical Coordinate**

The cylindrical coordinates (s, ϕ, z) of a point P are defined in Fig. 1.42. Notice that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P *from the z axis*, whereas the spherical coordinate r is the distance from the *origin*. The relation to Cartesian coordinates is

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z$$

The unit vectors

$$\begin{aligned} \hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned}$$

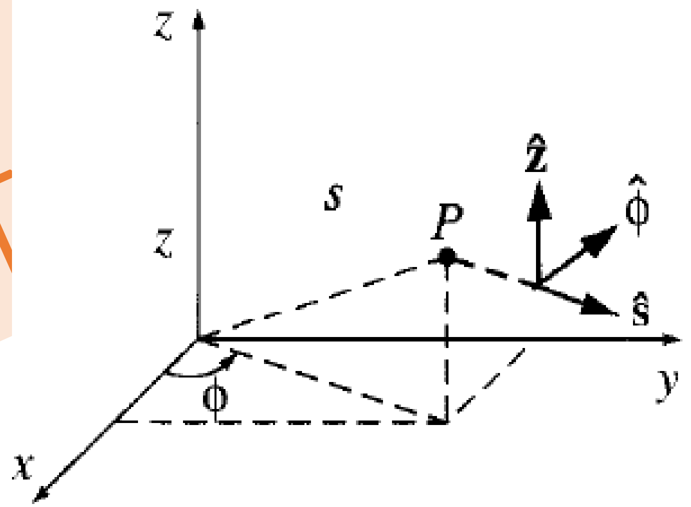


Figure 1.42

- **Infinitesimal Displacement**

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}},$$

• Differential Calculus: Ordinary Differential

Question: Suppose we have a function of one variable: $f(x)$. What does the derivative, df/dx, do for us? *Answer:* It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

$$df = \left(\frac{df}{dx}\right) dx. \qquad (1.33)$$

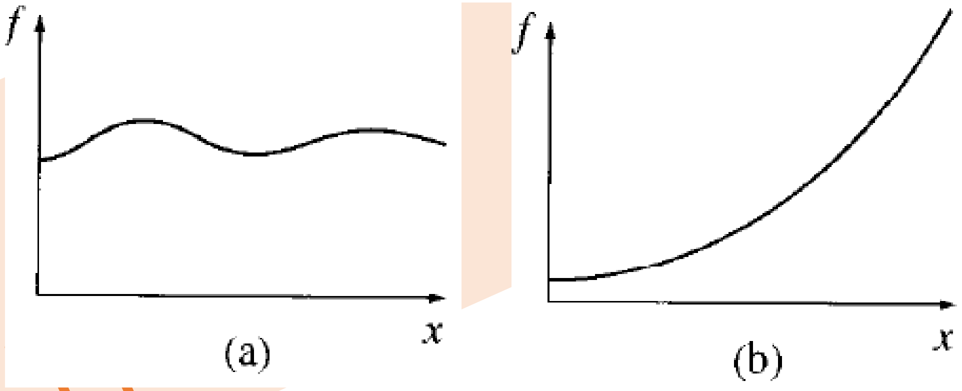


Figure 1.17

Geometrical Interpretation: The derivative df/dx is the *slope* of the graph of f versus x.

In words: If we change x by an amount dx, then f changes by an amount df; the derivative is the proportionality factor. For example, in Fig. 1.17(a), the function varies slowly with x, and the derivative is correspondingly small. In Fig. 1.17(b), f increases rapidly with x, and the derivative is large, as you move away from $x = 0$.

- **Gradient**

Suppose, now, that we have a function of *three* variables—say, the temperature $T(x, y, z)$ in a room. (Start out in one corner, and set up a system of axes; then for each point (x, y, z) in the room, T gives the temperature at that spot.) We want to generalize the notion of "derivative" to functions like T, which depend not on *one* but on *three* variables.

Now a derivative is supposed to tell us how fast the function varies, if we move a little distance. But this time the situation is more complicated, because it depends on what *direction* we move: If we go straight up, then the temperature will probably increase fairly rapidly, but if we move horizontally, it may not change much at all. In fact, the question "How fast does T vary?" has an infinite number of answers, one for each direction we might choose to explore.

A theorem on partial derivatives states

- If *T=T(x,y,z)* then

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz. \qquad (1.34)$$

- .

This tells us how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. Notice that we do *not* require an infinite number of derivatives—*three* will suffice: the *partial* derivatives along each of the three coordinate directions.

Equation 1.34 is reminiscent of a dot product:

$$dT = \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) = (\nabla T) \cdot (d\mathbf{l}), \quad (1.35)$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \quad (1.36)$$

is the **gradient** of T. ∇T is a *vector* quantity, with three components; it is the generalized derivative we have been looking for. Equation 1.35 is the three-dimensional version of Eq. 1.33.

- .

Geometrical Interpretation of the Gradient: Like any vector, the gradient has *magnitude* and *direction*. To determine its geometrical meaning, let's rewrite the dot product (1.35) in abstract form:

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}| \cos\theta, \tag{1.37}$$

where θ is the angle between ∇T and $d\mathbf{l}$. Now, if we *fix* the *magnitude* $|d\mathbf{l}|$ and search around in various *directions* (that is, vary θ), the *maximum* change in T evidentally occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $|d\mathbf{l}|$, dT is greatest when I move in the *same direction* as ∇T. Thus:

The gradient ∇T points in the direction of maximum increase of the function T.

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

• **Discussion**

Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent. That is the *direction* of the gradient. Now measure the *slope* in that direction (rise over run). That is the *magnitude* of the gradient. (Here the function we're talking about is the height of the hill, and the coordinates it depends on are positions—latitude and longitude, say. This function depends on only *two* variables, not *three*, but the geometrical meaning of the gradient is easier to grasp in two dimensions.) Notice from Eq. 1.37 that the direction of maximum *descent* is opposite to the direction of maximum *ascent*, while at right angles ($\theta = 90°$) the slope is zero (the gradient is perpendicular to the contour lines). You can conceive of surfaces that do not have these properties, but they always have "kinks" in them and correspond to nondifferentiable functions.

What would it mean for the gradient to vanish? If $\nabla T = 0$ at (x, y, z), then $dT = 0$ for small displacements about the point (x, y, z). This is, then, a **stationary point** of the function $T(x, y, z)$. It could be a maximum (a summit), a minimum (a valley), a saddle point (a pass), or a "shoulder." This is analogous to the situation for functions of *one* variable, where a vanishing derivative signals a maximum, a minimum, or an inflection. In particular, if you want to locate the extrema of a function of three variables, set its gradient equal to zero.

- **Ex** Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

Solution;

$$\begin{aligned}
\nabla r &= \frac{\partial r}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\,\hat{\mathbf{z}} \\
&= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{x}} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{y}} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{z}} \\
&= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.
\end{aligned}$$

Problem 1.11 Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

- Ex

Problem 1.13 Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

- **The Del Operator** ∇

The gradient has the formal appearance of a vector, ∇, "multiplying" a scalar T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) T. \qquad (1.38)$$

(For once I write the unit vectors to the *left*, just so no one will think this means $\partial \hat{\mathbf{x}}/\partial x$, and so on—which would be zero, since $\hat{\mathbf{x}}$ is constant.) The term in parentheses is called **"del"**:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \qquad (1.39)$$

Of course, del is *not* a vector, in the usual sense. Indeed, it is without specific meaning until we provide it with a function to act upon. Furthermore, it does not "multiply" T; rather, it is an instruction to *differentiate* what follows. To be precise, then, we should say that ∇ is a **vector operator** that *acts upon* T, not a vector that multiplies T.

- **Note**

Now an ordinary vector **A** can multiply in three ways:

1. Multiply a scalar a : $\mathbf{A}a$;
2. Multiply another vector **B**, via the dot product: $\mathbf{A} \cdot \mathbf{B}$;
3. Multiply another vector via the cross product: $\mathbf{A} \times \mathbf{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**);
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the **curl**).

- **The Divergence**

From the definition of ∇ we construct the divergence:

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \qquad (1.40)$$

Observe that the divergence of a vector function $\mathbf{v}$ is itself a *scalar* $\nabla \cdot \mathbf{v}$. (You can't have the divergence of a scalar: that's meaningless.)

Geometrical Interpretation: The name **divergence** is well chosen, for $\nabla \cdot \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ spreads out (diverges) from the point in question. For example, the vector function in Fig. 1.18a has a large (positive) divergence (if the arrows pointed *in*, it would be a large *negative* divergence), the function in Fig. 1.18b has zero divergence, and the function in Fig. 1.18c again has a positive divergence. (Please understand that $\mathbf{v}$ here is a *function*—there's a different vector associated with every point in space. In the diagrams,

- .Imagine that you are standing...

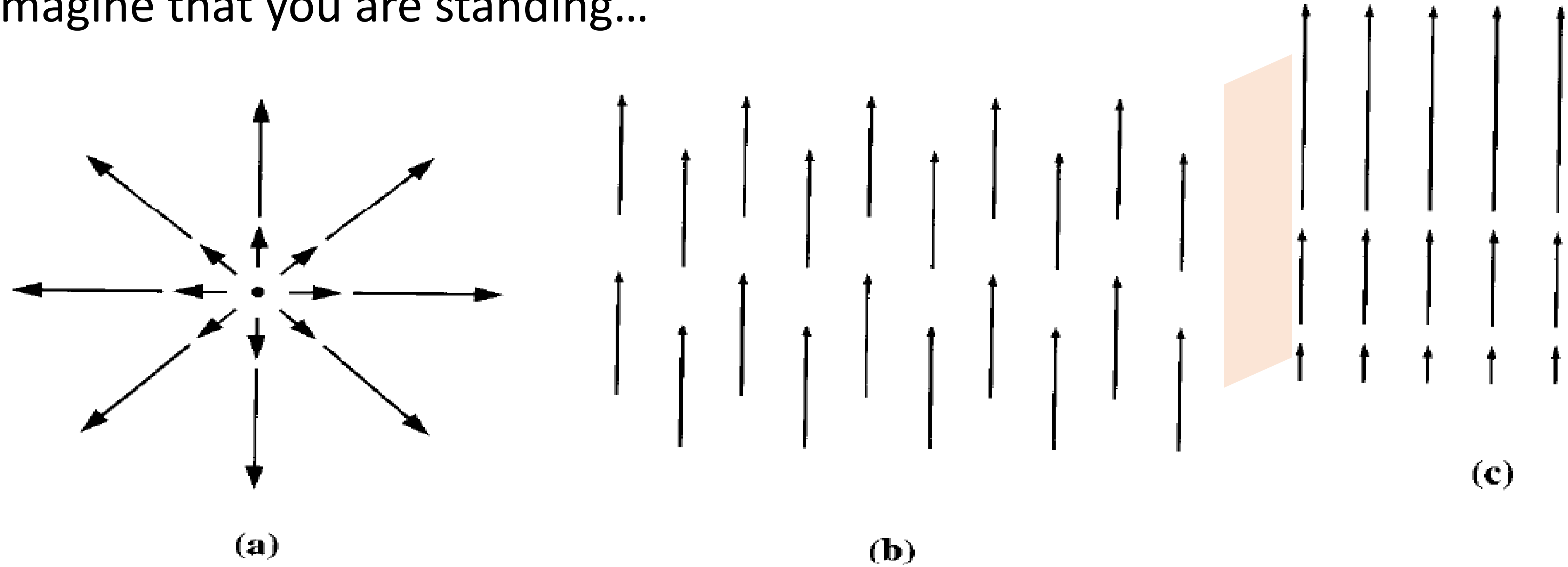


Figure 1.18

at the edge of a pond. Sprinkle some sawdust or pine needles on the surface. If the material spreads out, then you dropped it at a point of positive divergence; if it collects together, you dropped it at a point of negative divergence. (The vector function **v** in this model is the velocity of the water—this is a *two*-dimensional example, but it helps give one a “feel” for what the divergence means. A point of positive divergence is a source, or “faucet”; a point of negative divergence is a sink, or “drain.”)

- **Ex**

Suppose the functions in Fig. 1.18 are $\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{v}_b = \hat{\mathbf{z}}$, and $\mathbf{v}_c = z\,\hat{\mathbf{z}}$. Calculate their divergences.

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

this function has a positive divergence.

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0,$$

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1.$$

- HW

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

Problem 1.16 Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you... can you explain it?

- **The Curl**

From the definition of ∇ we construct the curl:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \quad (1.41)$$

Notice that the curl of a vector function $\mathbf{v}$ is, like any cross product, a *vector*. (You cannot have the curl of a scalar; that's meaningless.)

Geometrical Interpretation: The name **curl** is also well chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ "curls around" the point in question. Thus the three functions in Fig. 1.18 all have zero curl (as you can easily check for yourself), whereas the functions in Fig. 1.19 have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest. Imagine (again) you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero *curl*. A whirlpool would be a region of large curl.

- **Representation of Curl**

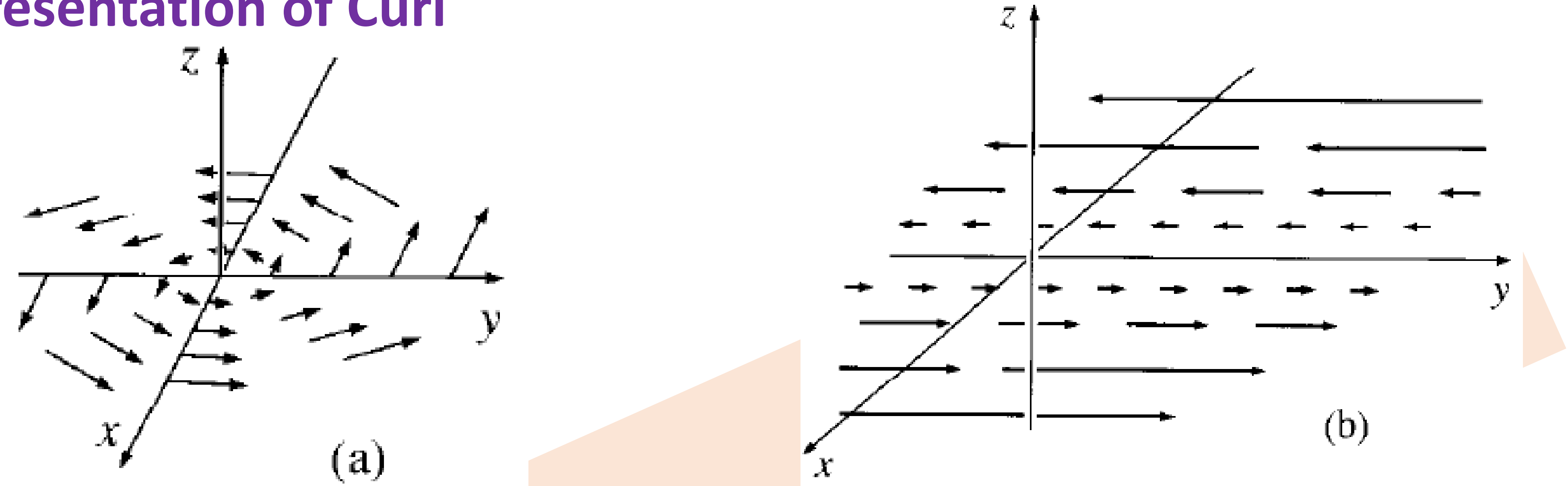


Figure 1.19

- **Note:**

We know that $\vec{V} = \vec{\omega} \times \vec{r}$, where ω is the angular velocity, $\vec{V}$ *is the linear velocity and* $\vec{r}$ *is the position vector of a point on the rotating body.*

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V}$$

$$\left[\begin{array}{l} \vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k} \\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \end{array}\right]$$

If Curl $\overline{F} = 0$, the field F is termed as *irrotational*.

- Ex

Suppose the function sketched in Fig. 1.19a is $\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, and that in Fig. 1.19b is $\mathbf{v}_b = x\hat{\mathbf{y}}$. Calculate their curls.

- Soln

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{z}},$$

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{\mathbf{z}}.$$

As expected, these curls point in the $+z$ direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out"... it just "curls around.")

- Notes

Prove that the divergence of a curl is always zero.

Prove that the curl of a gradient is always zero.

- **Curl & Div Theorems**

- If the curl of a vector field (**F**) vanishes (everywhere), then **F** can be written as the gradient of a **scalar potential** (V):

$$\nabla \times \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = -\nabla V.$$

Curl-less (or "**irrotational**") **fields**. The following conditions are equivalent (that is, **F** satisfies one if and only if it satisfies all the others):

(a) $\nabla \times \mathbf{F} = 0$ everywhere.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path, for any given end points.

(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

(d) **F** is the gradient of some scalar, $\mathbf{F} = -\nabla V$.

- **Theorem 2**

If the divergence of a vector field (**F**) vanishes (everywhere), then **F** can be expressed as the curl of a **vector potential** (**A**):

$$\nabla \cdot \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = \nabla \times \mathbf{A}.$$

Divergence-less (or "**solenoidal**") **fields.** The following conditions are equivalent:

(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.

(b) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface, for any given boundary line.

(c) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

(d) **F** is the curl of some vector, $\mathbf{F} = \nabla \times \mathbf{A}$.

- Ex *Show that* $\vec{A} = (6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ *is irrotational.*

Find ϕ *such that* $\vec{A} = \vec{\nabla}\phi$. *(DU, 2012)*

Solution. We have

$$\vec{A} = (6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k} \;;$$

$$\vec{\nabla}\times\vec{A} = \nabla\times[(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \hat{i}(-1+1)-\hat{j}(3z^2-3z^2)+\hat{k}(6x-6x) = \hat{i}(0)-\hat{j}(0)+\hat{k}(0) = 0$$

Hence, $\overline{A}$ is irrotational. $\Rightarrow \overline{A} = \overline{\nabla}\phi$, where ϕ is called scalar potential.

$$d\phi = \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz = \left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{i}\,dx+\hat{j}\,dy+\hat{k}dz) = \nabla\phi\cdot dr = A\,.\,dr$$

$$= [(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}]\cdot[\hat{i}dx+\hat{j}dy+\hat{k}dz]$$

$$= (6xy+z^3)dx+(3x^2-z)\,dy+(3xz^2-y)\,dz$$

$$= (6xy\,dx+3x^2\,dy)-(ydz+zdy)+(z^3dx+3xz^2dz)$$

$$\phi = \int(6xy\,dx+3x^2dy)-\int(y\,dz+zdy)+\int(z^3\,dx+3xz^2dz) = 3x^2y-yz+xz^3+C$$ **Ans.**

- Ex *For a solenoidal vector* $\vec{F}$*, show that curl curl curl curl* $\vec{F} = \nabla^4 \vec{F}$.

Solution. Since vector $\vec{F}$ is solenoidal, so div $\vec{F} = 0$ ... (1)

We know that curl curl $\vec{F}$ = grad div ($\vec{F} - \nabla^2 \vec{F}$) ... (2)

Using (1) in (2), grad div $\vec{F}$ = grad (0) = 0 ... (3)

On putting the value of grad div F in (2), we get

curl curl $\vec{F} = - \nabla^2 \vec{F}$ ... (4)

Now, curl curl curl curl $\vec{F}$ = curl curl $(- \nabla^2 \vec{F})$ [Using (4)]

$= -$ curl curl $(\nabla^2 \vec{F}) = -$ [grad div $(\nabla^2 \vec{F}) - \nabla^2 (\nabla^2 \vec{F})$] [Using (2)]

$= -$ grad $(\nabla \cdot \nabla^2 \vec{F}) + \nabla^2 (\nabla^2 \vec{F}) = -$ grad $(\nabla^2 \nabla \cdot \vec{F}) + \nabla^4 \vec{F}$ $[\nabla \cdot \vec{F} = 0]$

$= 0 + \nabla^4 \vec{F} = \nabla^4 \vec{F}$ [Using (1)] **Proved.**

- **Problems 1**

(a) Let $\mathbf{F}_1 = x^2\,\hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.

(b) Show that $\mathbf{F}_3 = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

- (c) *Find the divergence and curl of* $\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ *at* $(2, -1, 1)$
- (d) *If* $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of curl* $\vec{V}$.
- (e) *Prove that* $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ *is both solenoidal and irrotational.*

For solenoidal, we have to prove $\vec{\nabla}.\vec{F} = 0.$

For irrotational, we have to prove Curl $\vec{F} = 0.$

$\nabla \times \vec{F} = 0$

$\vec{F}$ is irrotational $\quad\therefore\quad F$ is conservative.

- SET:1 Divergence Problems

1. If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (*i*) div $\left(\dfrac{\vec{r}}{|\bar{r}|^3}\right) = 0$,

   (*ii*) div $(r\,\phi) = 3\phi + r$ grad ϕ.

2. Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

   (*DU, I Sem. 2012, R.G.P.V., Bhopal, Dec. 2003*)

3. Show that $\nabla.(\phi\, A) = \nabla\phi.A + \phi(\nabla.A)$
4. If ρ, ϕ, z are cylindrical coordinates, show that grad $(\log \rho)$ and grad ϕ are solenoidal vectors.
5. Obtain the expression for $\nabla^2 f$ in spherical coordinates from their corresponding expression in orthogonal curvilinear coordinates.
6. $\nabla.(\nabla\phi) = \nabla^2\phi$
7. $\vec{\nabla} \times \dfrac{(\vec{A} \times \vec{R})}{r^n} = \dfrac{(2-n)\vec{A}}{r^n} + \dfrac{n(\vec{A}.\vec{R})\vec{R}}{r^{n+2}}, r = |\vec{R}|$
8. div $(f\,\nabla\, g)$ – div $(g\,\nabla\, f) = f\,\nabla^2 g - g\,\nabla^2 f$

• SET:2 Curl Problems

1. Find the divergence and curl of the vector field $V = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$.

Ans. Divergence $= 4x$, Curl $= (2y - x)\hat{i} + y\hat{j} + 4y\hat{k}$

2. If a is constant vector and r is the radius vector, prove that

(i) $\nabla(\vec{a}.\vec{r}) = \vec{a}$ (ii) $\text{div}(\vec{r} \times \vec{a}) = 0$ (iii) $\text{curl}(\vec{r} \times \vec{a}) = -2\vec{a}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

3. Prove that:

$$\nabla(A.B) = (A.\nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$$

4. If $F = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that $F.\text{curl } F = 0$.

5. $\vec{\nabla} \times (\phi\vec{F}) = (\vec{\nabla}\phi) \times \vec{F} + \phi(\vec{\nabla} \times \vec{F})$

6. $\nabla.(\vec{F} \times \vec{G}) = \vec{G}.(\nabla \times \vec{F}) - \vec{F}.(\nabla \times \vec{G})$

7. Prove that curl $(\vec{a} \times \vec{r}) = 2a$

8. Prove that Div. (curl $\vec{v}$) $= \nabla.(\nabla \times \vec{v}) = 0$

9. If $V = e^{xyz}(\hat{i} + \hat{j} + \hat{k})$, find curl V **Ans.**

•

10. If $\vec{u} = \frac{\vec{r}}{r^2}$, then evaluate curl $\vec{u}$ **Ans.** 0

11. Evaluate curl grad, r^m, where $\vec{r} = |\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k})$ **Ans.** 0

12. Find div $\vec{F}$ and curl F where F = grad $(x^3 + y^3 + z^3 - 3xyz)$. *(R.G.P.V. Bhopal Dec. 2003)*

Ans. div $\vec{F} = 6(x + y + z)$, curl $\vec{F} = 0$

13. Find out values of a, b, c for which $\vec{v} = (x + y + az)\hat{i} + (bx + 3y - z)\hat{j} + (3x + cy + z)\hat{k}$ is irrotational.

Ans. $a = 3$, $b = 1$, $c = -1$

14. Determine the constants a, b, c, so that $\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational. Hence find the scalar potential ϕ such that $\vec{F}$ = grad ϕ. *(R.G.P.V. Bhopal, Feb. 2005)*

Ans. $a = 4$, $b = 2$, $c = 1$; Potential $\phi = \left(\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx\right)$

Choose the correct alternative:

15. The magnitude of the vector drawn in a direction perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point (1, –1, 2) is

(*i*) $\frac{2}{3}$ (*ii*) $\frac{3}{2}$ (*iii*) 3 (*iv*) 6 (*A.M.I.E.T.E., Summer 2000*) **Ans.** (*iv*)

16. If $u = x^2 - y^2 + z^2$ and $\bar{V} = x\hat{i} + y\hat{j} + z\hat{k}$ then $\nabla \cdot (u\bar{V})$ is equal to

(*i*) $5u$ (*ii*) $5|\vec{V}|$ (*iii*) $5(u - |\vec{V}|)$ (*iv*) $5(u - |\vec{V}|)$

(*A.M.I.E.T.E., June 2007*) **Ans.** (*i*)

17. A unit normal to $x^2 + y^2 + z^2 = 5$ at (0, 1, 2) is equal to

(*i*) $\frac{1}{\sqrt{5}}(\hat{i} + \hat{j} + \hat{k})$ (*ii*) $\frac{1}{\sqrt{5}}(\hat{i} + \hat{j} - \hat{k})$ (*iii*) $\frac{1}{\sqrt{5}}(\hat{j} + 2\hat{k})$ (*iv*) $\frac{1}{\sqrt{5}}(\hat{i} - \hat{j} + \hat{k})$

(*A.M.I.E.T.E., Dec. 2008*) **Ans.** (*iii*)

18. The directional derivative of $\phi = x\,y\,z$ at the point (1, 1, 1) in the direction $\hat{i}$ is:

(*i*) –1 (*ii*) $-\frac{1}{3}$ (*iii*) 1 (*iv*) $\frac{1}{3}$ **Ans.** (*iii*)

- Problems
- 1. Check the divergence of function

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

- 2. Check the curl of function

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}.$$