Differential Equation 2

- First & Second Order diff Equation
- Exact and Differential equations
- Non-exact Differential equations
- Different rules to solve Non-exact differential equations
- Homogeneous & Non Homogeneous Differential Equation (Operator method)

• EXACT DIFFERENTIAL EQUATION

An exact differential equation is formed by directly differentiating its primitive (solution) without any other process

$$Mdx + Ndy = 0$$

is said to be an exact differential equation if it satisfies the following condition $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

where $\frac{\partial M}{\partial y}$ denotes the differential co-efficient of M with respect to y keeping x constant and $\frac{\partial N}{\partial x}$,

Method for Solving Exact Differential Equations

Step I. Integrate M w.r.t. x keeping y constant

Step II. Integrate w.r.t. y, only those terms of N which do not contain x.

Step III. Result of I + Result of II = Constant.

- **EX1** Solve:

$$(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$$

Solution. Here, $M = 5x^4 + 3x^2y^2 - 2xy^3$, $\quad N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now $\int M\,dx + \int (\text{terms of } N \text{ is not containing } x)\,dy = C$

$$\int \left(5x^4 + 3x^2y^2 - 2xy^3\right)dx + \int -5y^4\,dy = C$$

$$\Rightarrow \quad x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

- **Ex 2** $Solve: \left\{2xy\cos x^2 - 2xy + 1\right\}dx + \left\{\sin x^2 - x^2 + 3\right\}dy = 0$

Solution. Here we have

$$\left\{2xy\cos x^2 - 2xy + 1\right\}dx + \left\{\sin x^2 - x^2 + 3\right\}dy = 0$$

$$M\,dx + N\,dy = 0$$

Comparing (1) and (2), we get

$$M = 2xy\cos x^2 - 2xy + 1 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$$

$$N = \sin x^2 - x^2 + 3 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$$

Here, $\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

So the given differential equation is exact differential equation.

Hence solution is $\displaystyle\int_{y \text{ as const}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

- ..

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int 3\,dy = C$$

$$\Rightarrow \quad \int [y(2x\cos x^2) - y(2x) + 1]\,dx + 3\int dy = C$$

$$\Rightarrow \quad y\int 2x\cos x^2 dx - y\int 2x\,dx + \int 1\,dx + 3\int y\,dy = C$$

Put $x^2 = t$ so that $2x\ dx = dt$

$$y\int \cos t\,dt - 2y\frac{x^2}{2} + x + 3y = C$$

$$\Rightarrow \quad y \sin t - x^2 y + x + 3y = C$$

$$y \sin x^2 - yx^2 + x + 3y = C$$

- **EXERCISE** Solve the following equations

1. $(x + y - 10)\, dx + (x - y - 2)\, dy = 0$ **Ans.** $\frac{x^2}{2} + xy - 10x - \frac{y^2}{2} - 2y = C$

2. $(y^2 - x^2)\, dx + 2x\, y\, dy = 0$ **Ans.** $\frac{x^3}{3} = x\, y^2 + C$

3. $\left(1 + 3e^{x/y}\right) dx + 3e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$ *(R.G.P.V. Bhopal, Winter 2010)* **Ans.** $x + 3y\, e^{x/y} = C$

4. $(2x - y)\, dx = (x - y)\, dy$ **Ans.** $xy = x^2 + \frac{y^2}{2} + C$

5. $(y \sec^2 x + \sec x \tan x)\, dx + (\tan x + 2y)\, dy = 0$ **Ans.** $y \tan x + \sec x + y^2 = C$

6. $(ax + hy + g)\, dx + (hx + by + f)\, dy = 0$ **Ans.** $ax^2 + 2h\, xy + by^2 + 2gx + 2fy + C = 0$

7. $(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y)\, dy = 0$ **Ans.** $\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = C$

8. $(2xy + e^y)\, dx + (x^2 + xe^y)\, dy = 0$ **Ans.** $x^2y + xe^y = C$

9. $(x^2 + 2ye^{2x})\, dy + (2xy + 2y^2e^{2x})\, dx = 0$ **Ans.** $x^2y + y^2\, e^{2x} = C$

10. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + (x + \log x - x \sin y)\, dy = 0$ *(M.D.U., 2010)*

Ans. $y\,(x + \log x) + x \cos y = C$

11. $(x^3 - 3xy^2)\, dx + (y^3 - 3x^3y)\, dy = 0,\ y(0) = 1$ **Ans.** $x^4 - 6x^2y^2 + y^4 = 1$

• Non-Exact Diff Equation: Equation Reducible to Exact Differential Eqn

Sometimes a differential equation which is not exact may become so, on multiplication by a suitable function known as the integrating factor.

Rule 1. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f(x)$, then $\text{I.F.} = e^{\int f(x)\,dx}$

• Ex *Solve* $(2x \log x - xy)\, dy + 2y\, dx = 0$

Solution. $M = 2y,\qquad N = 2x \log x - xy$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 2(1 + \log x) - y$$

Here, $$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2 - 2 - 2\log x + y}{2x \log x - xy} = \frac{-(2 \log x - y)}{x\,(2 \log x - y)} = -\frac{1}{x} = f(x)$$

$$\text{I.F.} = e^{\int f(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

On multiplying the given differential equation (1) by $\dfrac{1}{x}$, we get

$$\frac{2y}{x}dx + (2\log x - y)dy = 0$$

$$\int \frac{2y}{x}dx + \int -y\,dy = c$$

$$2y\log x - \frac{1}{2}y^2 = c$$

- EXERCISE

Solve the following differential equations:

1. $(y \log y)\, dx + (x - \log y)\, dy = 0$ **Ans.** $2x \log y = c + (\log y)^2$
2. $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}\left(1 + y^2\right)x\, dy = 0$ **Ans.** $\frac{yx^4}{4} + \frac{y^3x^4}{12} + \frac{x^6}{12} = c$
3. $(y - 2x^3)\, dx - x(1 - xy)\, dy = 0$ **Ans.** $-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$
4. $(x \sec^2 y - x^2 \cos y)\, dy = (\tan y - 3x^4)\, dx$ **Ans.** $-\frac{1}{x}\tan y - x^3 + \sin y = c$
5. $(x - y^2)\, dx + 2xy\, dy = 0$ **Ans.** $y^2 = cx - x \log x$

- **NON EXACT DIFFERENTIAL EQUATION**

Rule II. If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ is a function of y alone, say $f(y)$, then $\text{I.F.}=e^{\int f(y)dy}$

- **EX1** *Solve* $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0$

Solution. Here $M=y^4+2y$; $N=xy^3+2y^4-4x$

$$\therefore \quad \frac{\partial M}{\partial y}=4y^3+2; \qquad \frac{\partial N}{\partial x}=y^3-4$$

$$\therefore \quad \frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{(y^3-4)-(4y^3+2)}{y^4+2y}=\frac{-3(y^3+2)}{y(y^3+2)}=-\frac{3}{y}=f(y)$$

$$\text{I.F.}=e^{\int f(y)dy}=e^{\int -\frac{3}{y}dy}=e^{-3\log y}=e^{\log y^{-3}}=y^{-3}=\frac{1}{y^3}$$

On multiplying the given equation (1) by $\dfrac{1}{y^3}$ we get the exact differential equation.

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$$

$$\int\left(y+\frac{2}{y^2}\right)dx+\int 2y\;dy=c \quad\Rightarrow\quad x\left(y+\frac{2}{y^2}\right)+y^2=c$$

- **EXERCISE**: Solve the following

1. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$ **Ans.** $x^3y^2 + \dfrac{x^2}{y} = c$

2. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ **Ans.** $\dfrac{x^2y^4}{2} + xy^2 + \dfrac{y^6}{3} = c$

3. $y(x^2y + e^x)dx - e^x dy = 0$ **Ans.** $\dfrac{x^3}{3} + \dfrac{e^x}{y} = c$

4. $(2x^4y^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ **Ans.** $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$

- **NON EXACT DIFFERENTIAL EQUATION**

Rule III. If M is of the form $M = y\,f_1(xy)$ and N is of the form $N = x\,f_2(xy)$

Then $$\text{I.F.} = \frac{1}{M.x - N.y}$$

- **EX1** *Solve* $y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0$

Solution. $y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0$

Dividing (1) by xy, we get

$$y\,(1 + 2xy)\,dx + x\,(1 - xy)\,dy = 0$$

$$M = y\,f_1(xy), \quad N = x\,f_2(xy)$$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy(1+2xy) - xy(1-xy)} = \frac{1}{3x^2y^2}$$

On multiplying (2) by $\frac{1}{3x^2y^2}$, we have an exact differential equation

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \quad \Rightarrow \quad \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$$

$$\Rightarrow \quad -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c \qquad \Rightarrow \quad -\frac{1}{xy} + 2\log x - \log y = b$$

• EXERCISE

Solve the following differential equations:

1. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ **Ans.** $\log\left(\frac{x}{y}\right) - xy = A$

2. $y\,(1 + xy)\,dx + x(1 - xy)\,dy = 0$ **Ans.** $xy\log\left(\frac{y}{x}\right) = c\,xy - 1$

2. $y\,(1 + xy)\,dx + x\,(1 + xy + x^2y^2)\,dy = 0$ **Ans.** $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c$

4. $(xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0$ **Ans.** $y \cos xy = cx$

- **NON EXACT DIFFERENTIAL EQUATION: RULE IV**

If the given equation $M\,dx + N\,dy = 0$ is homogeneous equation and $Mx + Ny \neq 0$, then $\dfrac{1}{Mx+Ny}$ is an integrating factor.

- **EX1** *Solve* $\dfrac{dy}{dx} = \dfrac{x^3+y^3}{xy^2}$

Solution. $(x^3 + y^3)\,dx - (xy^2)\,dy = 0$

Here $M = x^3 + y^3, \qquad N = -xy^2$

$$\text{I.F.} = \frac{1}{Mx+Ny} = \frac{1}{x(x^3+y^3) - xy^2(y)} = \frac{1}{x^4}$$

Multiplying (1) by $\dfrac{1}{x^4}$ we get $\dfrac{1}{x^4}(x^3+y^3)dx + \dfrac{1}{x^4}(-xy^2)dy = 0$

$\Rightarrow \quad \left(\dfrac{1}{x} + \dfrac{y^3}{x^4}\right)dx - \dfrac{y^2}{x^3}dy = 0$, which is an exact differential equation.

$$\int\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx = c \qquad \Rightarrow \qquad \log x - \frac{y^3}{3x^3} = c$$

• EXERCISE

Solve the following differential equations:

1. $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **Ans.** $-\dfrac{x^3}{3y^3} + \log y = c$

2. $(y^3 - 3xy^2)\,dx + (2x^2y - xy^2)\,dy = 0$ **Ans.** $\dfrac{y}{x} + 3\log x - 2\log y = c$

3. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ **Ans.** $\dfrac{x}{y} - 2\log x + 3\log y = c$

4. $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$ **Ans.** $x^2y^4 - x^4y^2 = c$

- SECOND ORDER DIFFERENTIAL EQUATION

13.1 LINEAR DIFFERENTIAL EQUATIONS

If the degree of the dependent variable and all derivatives is one, such differential equations are called *linear differential equations e.g.*

(1) $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+6y=x^2+x+1$ (2) $2\dfrac{d^2x}{dt^2}-\dfrac{dx}{dt}-3x=f(t)$

13.2 NON LINEAR DIFFERENTIAL EQUATIONS

If the degree of the dependent variable and / or its derivatives are of greater than 1 such differential equations are called one-linear differential equations.

(1) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y^2=\sin x$ (2) $\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)^2+y^2=e^x$ (3) $\left(\dfrac{d^2x}{dt^2}\right)^2+\dfrac{dx}{dt}+x=f(t)$

The order of a differential equation is the highest order of the derivative involved. All the above differential equations are of second order.

Fourier and Laplace transforms are mathematical tools to solve the differential equations.

• . **LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENTS**

The general form of the linear differential equation of second order is

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P and Q are constants and R is a function of x or constant.

Differential operator. Symbol D stands for the operation of differential *i.e.*,

$$Dy = \frac{dy}{dx}, \quad D^2 y = \frac{d^2 y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R$ can be written in the operator form.

$$D^2y + P\,Dy + Q\,y = R \qquad \Rightarrow \qquad (D^2 + PD + Q)\,y = R$$

HOMOGENEOUS & NON-HOMOGENEOUS DE

- **DIMENSION OF SPACE OF SOLUTION**

A differential equation is said to have dimension k if the differential equation has k linearly independent solutions, $y_1, y_2 \ldots\ldots\; y_k$,

For example; $y'' - 5y' + 6y = 0$ has two dimensional solution e^{2x}, e^{3x}

$y''' - 6y'' + 11y' + 6y = 0$ has three dimensional independent solution e^{x}, e^{2x}, e^{3x}.

NON-HOMOGENEOUS

Consider the differential equation

$$y'' + P(x)y' + Q(x)y = F(x) \qquad \ldots(1)$$

(1) is said to be non-homogeneous if R.H.S. of (1) *i.e.*, $F(x) \neq 0$

HOMOGENEOUS

When $F(x) = 0$

Then (1) is said to be complementary homogeneous linear differential equation of (1).

• How to obtain Soln

Complete Solution = Complementary Function + Particular Integral.

$\Rightarrow$ $$\boxed{y = C.F. + P.I.}$$

METHOD FOR FINDING THE COMPLEMENTARY FUNCTION

(1) In finding the complementary function, R.H.S. of the given equation is replaced by zero.

(2) Let $y = C_1\, e^{mx}$ be the C.F. of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad ...(1)$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) then $C_1 e^{mx}\,(m^2 + Pm + Q) = 0$

$\Rightarrow$ $m^2 + Pm + Q = 0$. It is called **Auxiliary equation**.

(3) Solve the auxiliary equation :

• Solution of quadratic eqn

Case I : Roots, Real and Different. If m_1 and m_2 are the roots, then the C.F. is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case II : Roots, Real and Equal. If both the roots are m_1, m_1 then the C.F. is

$$y = (C_1 + C_2 x)\, e^{m_1 x}$$

Case III: Roots Imaginary. If the roots are $\alpha \pm i\beta$, then the solution will be

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = e^{\alpha x}.[C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$

$$= e^{\alpha x}[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$$

- Example $\quad$ Solve: $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 15y = 0.$

Solution. Given equation can be written as

$$(D^2 - 8D + 15)\, y = 0$$

Here auxiliary equation is $m^2 - 8m + 15 = 0$

$\Rightarrow \qquad (m - 3)(m - 5) = 0 \qquad \therefore\ m = 3, 5$

Hence, the required solution is

$$y = C_1 e^{3x} + C_2 e^{5x}$$

- Example $\quad$ Solve: $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$

Solution. Given equation can be written as

$$(D^2 - 6D + 9)\, y = 0$$

A.E. is $m^2 - 6m + 9 = 0 \quad \Rightarrow \quad (m - 3)^2 = 0 \quad \Rightarrow \quad m = 3, 3$

Hence, the required solution is

$$y = (C_1 + C_2 x)\, e^{3x}$$

- Example 3. *The general solution of the differential equation*

$$\frac{d^5y}{dx^5}-\frac{d^3y}{dx^3}=0 \text{ is given by}$$

Solution. Here, we have $\frac{d^5y}{dx^5}-\frac{d^3y}{dx^3}=0$

or $D^5y - D^3y = 0 \Rightarrow (D^5 - D^3)y = 0 \Rightarrow D^3(D^2 - 1)y = 0$

A.E. is $m^3(m^2 - 1) = 0 \Rightarrow m = 0, 0, 0, 1, -1$

Here the solution is

$$y=(C_1+C_2x+C_3x^2)+C_3e^x+C_5e^{-x}$$

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• **EX** Solve: $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$, $y = 2$ and $\frac{dy}{dx}=\frac{d^2y}{dx^2}$ when $x = 0$.

Solution. Here the auxiliary equation is

$$m^2 + 4m + 5 = 0$$

Its roots are $-2 \pm i$

The complementary function is

$$y = e^{-2x}(A\cos x + B\sin x) \qquad \text{...(1)}$$

On putting $y = 2$ and $x = 0$ in (1), we get

$$2 = A$$

On putting $A = 2$ in (1), we have

$$y = e^{-2x}[2\cos x + B\sin x] \qquad \text{...(2)}$$

On differentiating (2), we get

$$\frac{dy}{dx} = e^{-2x}[-2\sin x + B\cos x] - 2e^{-2x}[2\cos x + B\sin x]$$

$$= e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$

$$\frac{d^2y}{dx^2} = e^{-2x}[(-2B-2)\cos x - (B-4)\sin x] - 2e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$

$$= e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

But $\frac{dy}{dx}=\frac{d^2y}{dx^2}$

$$e^{-2x}[(-2B-2)\sin x + (B-4)\cos x] = e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

On putting $x = 0$, we get

$$B - 4 = -4B + 6 \quad \Rightarrow \quad B = 2$$

(2) becomes, $y = e^{-2x}[2\cos x + 2\sin x] \Rightarrow y = 2e^{-2x}[\sin x + \cos x]$

- **EXERCISE:** Solve the following differential equation

1. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ **Ans.** $y = C_1 e^x + C_2 e^{2x}$ **2.** $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 30y = 0$ **Ans.** $y = C_1 e^{5x} + C_2 e^{-6x}$

3. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$ **Ans.** $y = (C_1 + C_2 x)\, e^{4x}$

4. $\frac{d^2y}{dx^2} + \mu^2 y = 0$ **Ans.** $y = C_1 \cos \mu x + C_2 \sin \mu x$

5. $(D^2 + 2D + 2)\, y = 0,\; y(0) = 0,\; y'(0) = 1$ **Ans.** $y = e^{-x} \sin x$ *(A.M.I.E.T.E., June 2006)*

6. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$ **Ans.** $y = C_1 e^{2x} + C_2 \cos 2x + C_3 \sin 2x$

7. $\frac{d^4y}{dx^4} - 32\frac{d^2y}{dx^2} + 256 = 0$ *(A.M.I.E.T.E., Dec. 2004)* **Ans.** $y = (C_1 + x) \cos 4x + (C^3 + C^4 x) \sin 4x$

8. $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$ **Ans.** $y = e^x [(C_1 + C_2 x) \cos x + (C_3 + C_4 x) \sin x]$

- **Rules to find the particular Integral**

(i) $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ If $f(a) = 0$, then the above rule fails.

Then $\frac{1}{f(D)}e^{ax} = x \cdot \frac{1}{f'(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax} \Rightarrow$ $\boxed{\frac{1}{f(D)}e^{ax} = x \cdot \frac{1}{f'(a)}e^{ax}}$

If $f'(a) = 0$ then $\boxed{\frac{1}{f(D)}e^{ax} = x^2 \frac{1}{f''(a)}e^{ax}}$

- **EX** $Solve : \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$

Solution. $(D^2 + 6D + 9)y = 5e^{3x}$

Auxiliary equation is $m^2 + 6m + 9 = 0 \Rightarrow (m + 3)^2 = 0 \Rightarrow m = -3, -3,$

$$\text{C.F.} = (C_1 + C_2x)\, e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2 + 6(3) + 9} = \frac{5e^{3x}}{36}$$

The complete solution is $y = (C_1 + C_2x)e^{-3x} + \frac{5e^{3x}}{36}$

- **Rule II**

$$\frac{1}{f(D)}x^n = [f(D)]^{-1}x^n.$$

Expand $[f(D)]^{-1}$ by the Binomial theorem in ascending powers of D as far as the result of operation on x^n is zero.

- **Rule III**

$$\frac{1}{f(D^2)}\sin ax = \frac{\sin ax}{f(-a^2)} \qquad \frac{1}{f(D^2)}\cdot\cos ax = \frac{\cos ax}{f(-a^2)}$$

- **Rule IV**

$$\frac{1}{f(D)}.e^{ax}\cdot\phi(x) = e^{ax}.\frac{1}{f(D+a)}.\phi(x)$$

- **Ex** $\quad Solve\ (D^2+6D+9)y = \dfrac{e^{-3x}}{x^3}.$

Solution A.E. is $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \quad \therefore \quad m = -3, -3$

C.F. $= (C_1 + C_2x)\, e^{-3x}$

$$\text{P.I.} = \frac{1}{D^2+6D+9}\frac{e^{-3x}}{x^3} = e^{-3x}\frac{1}{(D-3)^2+6(D-3)+9}\frac{1}{x^3} = e^{-3x}\frac{1}{D^2-6D+9+6D-18+9}\frac{1}{x^3} = e^{-3x}\frac{1}{D^2}(x^{-3})$$

$$= e^{-3x}\frac{1}{D}\left(\frac{x^{-2}}{-2}\right) = e^{-3x}\frac{x^{-1}}{(-2)(-1)} = \frac{e^{-3x}x^{-1}}{2} = \frac{e^{-3x}}{2x}$$

Hence, the solution is $y = (C_1 + C_2x)e^{-3x} + \dfrac{e^{-3x}}{2x}$

- **EX** $Solve : (D^2 - 4D + 4)\, y = x^3\, e^{2x}$

Solution. $(D^2 - 4D + 4)\, y = x^3\, e^{2x}$

A.E. is $m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)^2 = 0 \quad \Rightarrow \quad m = 2, 2$

C.F. $= (C_1 + C_2\, x)\, e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} x^3 \cdot e^{2x} = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x} \frac{1}{D^2} x^3 = e^{2x} . \frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x} \cdot \frac{x^5}{20}$$

The complete solution is $y = (C_1 + C_2 x)\, e^{2x} + e^{2x} . \dfrac{x^5}{20}$

- **EX** $$Solve:\ \frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$$

Solution. The given equation is $(D^2-2D+1)\,y = xe^x \sin x$

A.E. is $m^2-2m+1=0$ $\quad\therefore\ m = 1, 1$

C.F. $= (C_1 + C_2x)\,e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2}e^x.x\sin x = e^x\frac{1}{(D+1-1)^2}x\sin x = e^x\frac{1}{D^2}x\sin x = e^x.\frac{1}{D}\int x\sin x\,dx$$

Integrating by parts

$$= e^x\frac{1}{D}[x(-\cos x)-\int(-\cos x)\,dx] = e^x.\frac{1}{D}(-x\cos x+\sin x)$$

$$= e^x\int(-x\cos x+\sin x)\,dx = e^x\left\{-x\sin x+\int 1.\sin x\,dx-\cos x\right\}$$

$$= e^x[-x\sin x-\cos x-\cos x] = -e^x(x\sin x+2\cos x)$$

Hence, the complete solution is

$$y = (C_1 + C_2x)\,e^x - e^x(x\sin x + 2\cos x).$$ **Ans.**

- **EXERCISE**

1. $(D^2 - 5D + 6)\, y = e^x \sin x$ **Ans.** $y = C_1 e^{2x} + C_2 e^{3x} + \frac{e}{10}(3\cos x + \sin x)$

2. $\frac{d^2 y}{dx^2} - 7\frac{dy}{dx} + 10y = e^{2x} \sin x$ **Ans.** $y = C_1 e^{2x} + C_2 e^{5x} + \frac{e^{2x}}{10}(3\cos x - \sin x)$

3. $\frac{d^3 y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x$ **Ans.** $y = C_1 e^{-2x} + e^x (C_2 \cos x + C_3 \sin x) + \frac{xe^x}{20}(3\sin x - \cos x)$

4. $(D^2 - 4D + 3)\, y = 2xe^{3x} + 3e^{3x} \cos 2x$

Ans. $y = C_1 e^x + C_2 e^{3x} + \frac{1}{2} e^{3x}(x^2 - x) + \frac{3}{8} e^{3x}(\sin 2x - \cos 2x)$

5. $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}$ **Ans.** $y = (C_1 + C_2 x)\, e^{-x} - e^{-x} \log x$

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